Localized waves in FitzHugh-Nagumo equations

Chao-Nien Chen

National Tsing Hua University

Joint work with

Y.S. Choi

University of Connecticut

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A. M. Turing, Phil. Trans. R. Soc. Lond. 1952
The chemical basis of morphogenesis.

A fascinating idea proposed by Turing demonstrated that in a homogeneous medium, spatially heterogeneous distributed patterns can be produced from chemical reaction of two substances with different diffusivities.

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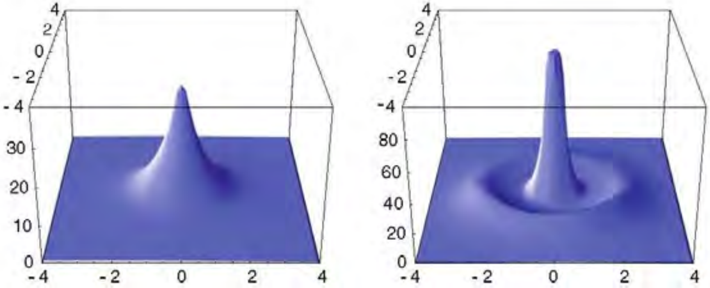
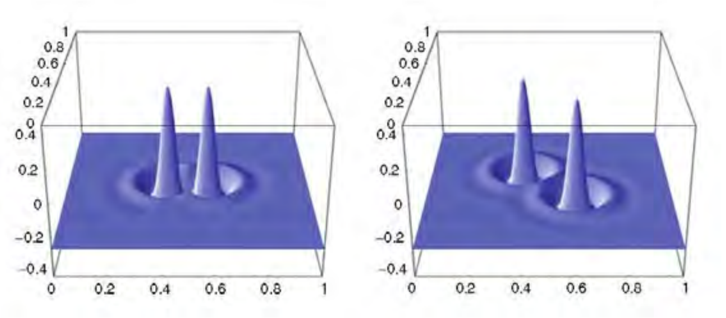
- Pulses are self-organized patterns with profiles that are in close proximity to a trivial background state except for one or several localized spatial regions where changes are substantial.
- Localized structures represent states which are far away from the homogeneous equilibrium.

Particle-like structures are commonly observed in physical, chemical and biological systems.





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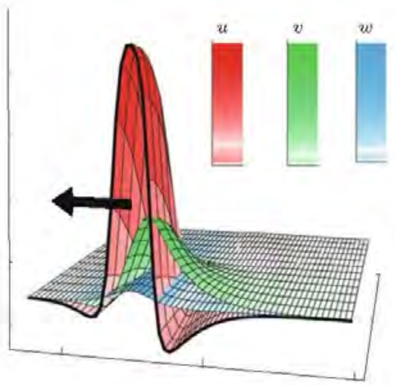
Depending on the system parameters and initial conditions, localized dissipative structures may stay at rest or propagate with a dynamically stabilized velocity. (Localized waves)



A well-known reaction-diffusion model for studying diffusion -induced instability and emergence of patterns is the system of FitzHugh-Nagumo equations:

$$\begin{aligned} u_t - \Delta u &= \frac{1}{d}(f(u) - v) \qquad u: \text{ activator} \\ v_t - \Delta v &= u - \gamma v \qquad v: \text{ inhibitor} \end{aligned} \tag{FN}$$

$$\gamma > 0 \qquad\qquad f: \text{cubic polynomial}$$

Suppose $(u(x-ct), v(x-ct))$ satisfies (FN), then when viewed by someone moving with the speed c, this solution keeps the same profile. Such a solution is referred to as a traveling wave, while it is a stationary pattern or standing wave if $c = 0$.

We may consider the case $c > 0$ only; for otherwise, reverse the direction of wave motion will do. Recall that

$$u_t = u_{xx} + f(u) - v$$

$$v_t = \varepsilon(u - \gamma v)$$

is a simplified model for the Hodgkin-Huxley system to describe the electrical impulses in the axon of the squid. Here $(u, v) = (0, 0)$ is the rest state, and the nerve impulse is generated by a finite excitation; the homogeneous ground state relaxes the characteristic shape of the pulse. For $\varepsilon << 1$, the existence of a traveling pulse has been treated as a singular perturbation problem in which the pulse is constructed by stitching together solutions of certain reduced systems.

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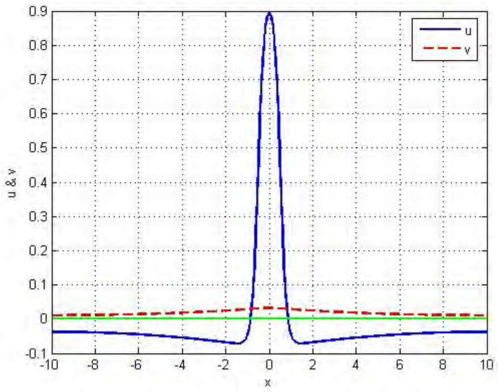
Theorem. Let $\beta \in (0, 1/2)$ be given.

(i) There exist $\hat{\gamma} > 0$ and $\hat{d} = \hat{d}(\gamma) > 0$ such that if $\gamma < \hat{\gamma}$ and $d < \hat{d}$, then there is a standing pulse solution (u, v) of (FN).

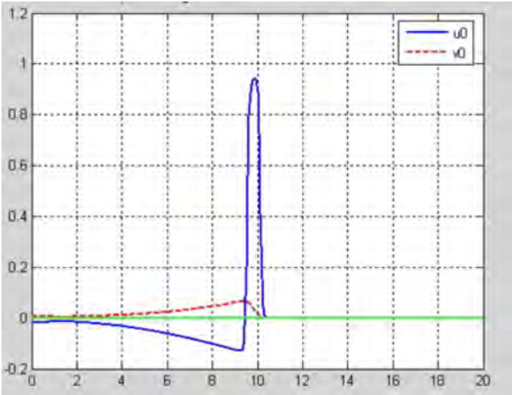
(ii) Both u and v are even functions on $(-\infty, \infty)$ and satisfy $(u, v) \to (0, 0)$ as $x \to \infty$.

(iii) u changes signs exactly once in $(0, \infty)$ while $v > 0$ and $v' < 0$ on $(0, \infty)$.

The profile of a standing pulse



The profile of a traveling pulse



There have been many interesting works on the traveling wave solutions for scalar reaction-diffusion equation

$$u_t = \Delta u + f(u).$$

A planar traveling wave satisfies an ordinary differential equation; the existence question has been studied by various methods. Lucia, Muratov and Novaga considered the functional

$$I_c[u] = \int_{\mathbf{R}} (\frac{1}{2}u_x^2 + F(u))e^{cx}dx,$$

where $F(u) = -\int_0^u f(s)ds$. Since $I_c(w(\cdot - a)) = e^a I_c(w)$, they seek a minimizer u_c of I_c under the constraint $\int_{\mathbf{R}} u_x^2 e^{cx} dx = 1$. When $I_c[u_c] \leq 0$, letting $\hat{c} = c\sqrt{1 - I_c[u_c]}$ and $\hat{u}(x) = u_c(\hat{c}x/c)$, they showed that $\hat{u}$ is a travelling wave with speed $\hat{c}$.

Heinze considered a different type of ansatz $u(c(\xi - ct), y)$ in the change of variables. Then a function satisfying

$$c^2(u_{xx} + u_x) + \Delta_y u + f(u) = 0$$

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Let $\mathbf{H}$ denote the Hilbert space equipped with the norm $||u||_{\mathbf{H}} = \int_\Omega (u_x^2 + |\nabla_y u|^2 + u^2) e^x dxdy$, where Ω is a cylinder. Heinze viewed a traveling wave solution $\hat{u}$ as a minimizer of $\int_\Omega \frac{1}{2} u_x^2 e^x dxdy$ subject to the constraint $\{u \in \mathbf{H} : \int_\Omega [\frac{1}{2}|\nabla_y u|^2 + F(u)] e^x dxdy = -1\}$. A stimulating result of this approach is that the Lagrange multiplier is nothing but $\frac{1}{c^2}$, where $|c|$ gives the wave speed.

Following the ansatz proposed by Heinze, we study the homoclinic solutions of

$$\begin{cases} dc^2 u_{xx} + dc^2 u_x + f(u) - v &= 0, \\ \\ c^2 v_{xx} + c^2 v_x + u - \gamma v &= 0. \end{cases} \qquad (1)$$

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Indeed if (u, v) satisfies (1) for some $c > 0$, there is a traveling wave to (FN). In particular, if $(u, v) \to (0, 0)$ as $|x| \to \infty$ then this is a traveling pulse solution.

Let $L^2_{ex} = \{u : \int_{-\infty}^{\infty} e^x (u(x))^2 \, dx < \infty\}$ be a Hilbert space equipped with a weighted norm $\|u\|_{L^2_{ex}} \equiv \sqrt{\int_{-\infty}^{\infty} e^x \, u^2 \, dx}$.
The Green function G of the differential operator $(\gamma - c^2 \frac{d^2}{dx^2} - c^2 \frac{d}{dx})$ is a positive function given by

$$G(x,s) = \begin{cases} \frac{1}{c\sqrt{c^2+4\gamma}} e^{-r_2 s} \, e^{r_2 x}, & \text{if } x < s, \\ \\ \frac{1}{c\sqrt{c^2+4\gamma}} e^{-r_1 s} \, e^{r_1 x}, & \text{if } x > s. \end{cases}$$

Here the solutions of the characteristic equation $c^2 r^2 + c^2 r - \gamma = 0$ are $\frac{1}{2c}(-c \pm \sqrt{c^2 + 4\gamma})$, denoted by r_1 and r_2 with $r_1 < -1 < 0 < r_2$.

$\mathcal{L}_c : L^2_{ex} \to L^2_{ex}$, since $1 + 2r_1 < 0$ and $1 + 2r_2 > 0$.

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$$\begin{aligned}
\int_{\mathbf{R}} e^x u_1(x)\, \mathcal{L}_c u_2\,(x)\, dx &= \int_{\mathbf{R}}\int_{\mathbf{R}} e^x e^s g(x,s) u_1(x) u_2(s)\, ds\, dx \\
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$$\mathcal{L}_c u\,(x) = \frac{e^{r_1 x}}{c\sqrt{c^2+4\gamma}} \int_{-\infty}^{x} e^{-r_1 s}\ u(s)\, ds + \frac{e^{r_2 x}}{c\sqrt{c^2+4\gamma}} \int_{x}^{\infty} e^{-r_2 s}\ u(s)\, ds$$

Define $F(\xi) = -\int_0^\xi f(\eta)\, d\eta = \xi^4/4 - (1+\beta)\xi^3/3 + \beta\xi^2/2$.
Let H^1_{ex} be a Hilbert space equipped with the norm

$$\|w\|_{H^1_{ex}} = \sqrt{\int_{\mathbf{R}} e^x w_x^2\, dx + \int_{\mathbf{R}} e^x w^2\, dx}\ .$$

Consider a functional $J_c : H^1_{ex} \to \mathbf{R}$ defined by

$$J_c(w) \equiv \int_{\mathbf{R}} e^x \{ \frac{dc^2}{2} w_x^2 + \frac{1}{2} w\, \mathcal{L}_c w + F(w) \}\, dx\ .$$

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- If $a \in \mathbf{R}$ and $w \in L^2_{ex}$ then $\mathcal{L}_c(w(\cdot - a)) = (\mathcal{L}_c w)(\cdot - a)$.
- For any $a \in \mathbf{R}$ and $w \in H^1_{ex}$, $J_c(w(\cdot - a)) = e^a J_c(w)$.

If $w \in H^1_{ex}(\mathbf{R})$ then $$\begin{cases} \frac{1}{4}\int_{\mathbf{R}} e^x w^2 \, dx & \leq \int_{\mathbf{R}} e^x w_x^2 \, dx \ , \\ e^x w^2(x) & \leq \int_x^\infty e^y w_y^2 \, dy \ . \end{cases}$$

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- $\mathcal{L}_c(L^2_{ex}) \subset H^1_{ex}$. In fact

$$\|\mathcal{L}_c u\|_{H^1_{ex}} \leq \frac{2\sqrt{5}}{c^2}\|u\|_{L^2_{ex}} \ .$$

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- For any $w \in H^1_{ex}$,

$$\int_{\mathbf{R}} e^x w_x^2\,dx \leq \|w\|^2_{H^1_{ex}} \leq 5\int_{\mathbf{R}} e^{cx} w_x^2\,dx;$$

in other words, $\|w_x\|_{L^2_{ex}}$ is an equivalent norm for H^1_{ex}.

$$
\begin{cases}
\|v\|_{L^2_{ex}} \le \frac{4}{c^2}\|u\|_{L^2_{ex}} \ , \\[2ex]
\|v'\|_{L^2_{ex}} \le \frac{2}{c^2}\|u\|_{L^2_{ex}} \ , \\[2ex]
0 \le \int_{\mathbf{R}} e^x \{c^2(\mathcal{L}_c u)'^2 + \gamma(\mathcal{L}_c u)^2\}\, dx = \int_{\mathbf{R}} e^x u\, \mathcal{L}_c u\, dx \ .
\end{cases}
$$

Let

$$
\mathcal{A} \equiv \{w \in H^1_{ex} : \int_{\mathbf{R}} e^x w_x^2\, dx = 2.\}.
$$

To seek traveling pulse solutions, we begin with studying the functional $J_c : \mathcal{A} \to \mathbf{R}$. Set $\mathcal{J}(c) \equiv \inf_{w\in\mathcal{A}} J_c(w)$.

Lemma. There exists a $\bar{c} = \bar{c}(d, \beta) > 0$ such that if $c \geq \bar{c}$ then $\mathcal{J}(c) > 0$.

Proof. For $w \in \mathcal{A}$, since the nonlocal term is non-negative,

$$\begin{aligned} J_c(w) &\geq \frac{dc^2}{2} \int_{\mathbf{R}} e^x w_x^2 \, dx + \int_{\mathbf{R}} e^x F(w) \, dx \\ &\geq dc^2 - M_2 \int_{\mathbf{R}} e^x w^2 \, dx \\ &\geq dc^2 - 8M_2 \\ &\geq 8M_2 \,. \end{aligned}$$

Clearly $\mathcal{J}(c) > 0$ if $\bar{c} = 4\sqrt{M_2/d}$.

The next step is to show that $\mathcal{J}(c) < 0$ when c is small. Let $y = x/c$ and $\tilde{\mathcal{L}}_c = (\gamma - \frac{d^2}{dy^2} - c\frac{d}{dy})^{-1}$. For $u \in \mathcal{A}$, if $\tilde{u}(y) \equiv u(x)$, it is easy to check that $\tilde{\mathcal{L}}_c\tilde{u}\,(y) = \mathcal{L}_c u(x)$, so is $\tilde{v}(y) = \tilde{\mathcal{L}}_c\tilde{u}(y)$ Clearly $u \in H^1_{ex}$ if and only if $\tilde{u} \in H^1_c(\mathbf{R}) = \{w : \int_{\mathbf{R}} e^{cy} w_y^2\, dy < \infty\}$. Similarly $\mathcal{L}_c u \in H^1_{ex}$ if and only if $\tilde{\mathcal{L}}_c\tilde{u} \in H^1_c$. A direct calculation gives

$$
\begin{aligned}
J_c(u) &= \int_{\mathbf{R}} e^x \{ \frac{dc^2}{2}(u'(x))^2 + F(u(x)) + \frac{1}{2}u(x)\,\mathcal{L}_c u\,(x)\}\, dx \\
&= c\int_{\mathbf{R}} e^{cy} \{ \frac{d}{2}(\tilde{u}'(y))^2 + F(\tilde{u}(y)) + \frac{1}{2}\tilde{u}(y)\,\tilde{\mathcal{L}}_c\tilde{u}\,(y)\}\, dy\,.
\end{aligned}
$$

Consider a piecewise linear function defined by

$$q_0(y) \equiv \begin{cases} 1, & \text{if } 0 \le y \le a, \\ \frac{(b-y)}{b-a}, & \text{if } a \le y \le b, \\ 0, & \text{if } y \ge b. \end{cases}$$

With an even extension, q_0 has a compact support on the real line. It has been shown that there is a $k_1 > 0$ such that

$$\int_{\mathbf{R}} \{ \frac{d}{2} q_0'^2 + F(q_0) + \frac{1}{2} q_0 \, \tilde{\mathcal{L}}_0 q_0 \, \} \, dy < 0$$

if $d \le k_1 \gamma$.

Lemma. Let $c \geq 0$ and $\{c_n\}_{n=1}^{\infty} \subset [0, \infty)$ such that $|c_n - c| \leq \min\{1, \gamma\}$ and $c_n \to c$ if $n \to \infty$. Suppose that there exists a $\delta > 0$ such that $(c-\delta)^+ \leq c_n \leq c + \delta$. If $w \in H_s^1$ for all $(c-\delta)^+ \leq s \leq c + \delta$, then

$$\|\tilde{\mathcal{L}}_{c_n} w - \tilde{\mathcal{L}}_c w\|_{H_c^1} \leq \frac{2(1+\gamma)}{\gamma^2} |c_n - c| \, \|w\|_{L_c^2} \, .$$

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Lemma. Let $c > 0$ and $\{c_n\}_{n=1}^{\infty} \subset (c/\sqrt{2}, c\sqrt{3}/\sqrt{2})$ such that $c_n \to c$ as $n \to \infty$. Then there exists a $M_4 > 0$ such that

$$\|\mathcal{L}_{c_n} w - \mathcal{L}_c w\|_{H_{ex}^1} \leq M_4 \, |c_n^2 - c^2| \, \|w\|_{L_{ex}^2}$$

for all $w \in H_{ex}^1$. Here M_4 is continuous in γ and c, and bounded if c is bounded away from zero.

Lemma. The function $\mathcal{J} : [\underline{c}, \bar{c}] \to \mathbf{R}$ is continuous.

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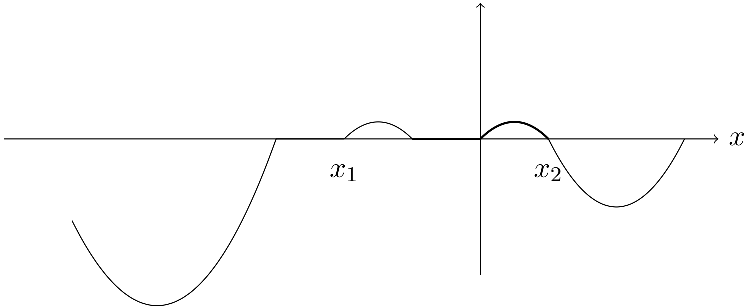
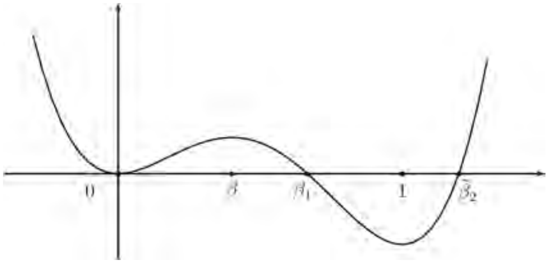
A Poincare type inequality shows that this solution decays to $(0, 0)$ at $+\infty$. However, it is more delicate and usually requires more efforts to investigate the asymptotic behavior of a traveling wave near $-\infty$ when such a solution is obtained from a weighted function space like H^1_{ex} via a variational approach In some situations, for instance in case of scalar reaction-diffusion equations, the maximum principle provides a help to complete this task.

We minimize J_c over the set of admissible functions in the class $-/+/-$. Roughly speaking, this is a topological constraint which requires that the functions in the admissible set change sign at most twice.



Let $\beta < \beta_1 < 1 < \beta_2$ satisfy $F(\beta_1) = F(\beta_2) = 0$. A continuous function w is in the class $-/+/-$, if there exist $-\infty \leq x_1 \leq x_2 \leq \infty$ such that $w \leq 0$ on $(-\infty, x_1] \cup [x_2, \infty)$, and $w \geq 0$ on $[x_1, x_2]$. The choice of x_1, x_2 is not necessarily unique. For instance, if $x_1 = -\infty$ and $x_2 = \infty$, then $w \geq 0$ on the real line. In case $x_1 = x_2 = \infty$, then $w \leq 0$ on the real line. Both examples are included in the class $-/+/-$.



The qualitative properties of the global minimizer u_0 can be accessed by arguing indirectly and performing step by step either local or global surgeries on u_0 to generate a new function $u_{new} \in \mathcal{A}$ with $J_c(u_{new}) < J_c(u_0)$.

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Lemma. Suppose that $u_{new} \in H^1_{ex}$, a function after making some changes on u_0. Then the change in the nonlocal term is

$$\int_{\mathbf{R}} e^x (u_{new}\,\mathcal{L}_c u_{new} - u_0\,\mathcal{L}_c u_0) = \int_{\mathbf{R}} e^x (u_{new} - u_0)\,\mathcal{L}_c(u_{new} + u_0)\ .$$

Proof. It is a direct consequence of the fact that $\mathcal{L}_c$ is self adjoint with respect to the L^2_{ex} inner product.

Lemma. Suppose that $u_{new} \in H^1_{ex}$, a function after making some changes on u_0. Then the change in the nonlocal term is

$$\int_{\mathbf{R}} e^x (u_{new}\, \mathcal{L}_c u_{new} - u_0\, \mathcal{L}_c u_0) = \int_{\mathbf{R}} e^x (u_{new} - u_0)\, \mathcal{L}_c (u_{new} + u_0) \ .$$

Proof. It is a direct consequence of the fact that $\mathcal{L}_c$ is self adjoint with respect to the L^2_{ex} inner product.

Remark. If the support of $u_{new} - u$ lies inside a finite interval $[a, b]$, then the change in the nonlocal term can be calculated within the same interval $[a, b]$. When $u_{new} - u$ is small, $\mathcal{L}_c u_{new}$ is close to $\mathcal{L}_c u$ on $[a, b]$, even though the decay behavior and the sign of $\mathcal{L}_c u_{new}$ near infinity can differ from those of $\mathcal{L}_c u$.

Let u_0 be a minimizer. The next lemma enables us to eliminate the possibility that a sharp corner appears on the graph of u_0.

Corner Lemma If x_0 and $\ell > 0$ are numbers such that $u_0(x_0) = 0$ and $u_0 \in C^1[x_0 - \ell, x_0] \cap C^1[x_0, x_0 + \ell]$, then $\lim_{x \to x_0^-} u_0'(x) = \lim_{x_0 \to x_0^+} u_0'(x)$.

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The inherited technical difficulty associated with the topological class $\mathcal{A}$ is partly alleviated.

The idea for the proof of corner lemma.

We argue indirectly. Suppose $u_0(x_0) = 0$, $\lim_{x \to x_0^-} u_0'(x) = a_1$ and $\lim_{x_0 \to x_0^+} u_0'(x) = a_2$ with $a_1 \neq a_2$. If u_0 were straight lines on either side of x_0, then

$$u_0(x) = \begin{cases} a_1(x - x_0)\ , & \text{if } x_0 - \ell \leq x \leq x_0, \\ a_2(x - x_0)\ , & \text{if } x_0 \leq x \leq x_0 + \ell. \end{cases}$$

If the points $(x_0 - \ell, u_0(x_0 - \ell))$ and $(x_0 + \ell, u_0(x_0 + \ell))$ on the graph of u_0 is joined by a straight line, the slope of this line is $(a_1 + a_2)/2$. This simple example gives a basic idea in the proof.

For a general C^1 function u_0, if $\ell_1 << \ell$, then $u_0'(x) = a_1 + o(1)$ for $x_0 - \ell_1 \le x \le x_0$; $u_0'(x) = a_2 + o(1)$ for $x_0 \le x \le x_0 + \ell_1$; and the straight line $y = L_1(x)$ joining $(x_0 - \ell_1, u_0(x_0 - \ell_1))$ and $(x_0 + \ell_1, u_0(x_0 + \ell_1))$ has a slope of $(a_1 + a_2)/2 + o(1)$. Set

$$u_{new}(x) = \begin{cases} u_0(x), & \text{if } x \le x_0 - \ell_1, \\ L_1(x), & \text{if } x_0 - \ell_1 \le x \le x_0 + \ell_1, \\ u_0(x), & \text{if } x \ge x_0 + \ell_1. \end{cases}$$

Then

$$\begin{aligned} & \frac{d}{2} \int_{x_0-\ell_1}^{x_0+\ell_1} \left\{ (u_{new})_x^2 - (u_0)_x^2 \right\} e^x dx \\ = \quad & \frac{d}{2} \left\{ \left(\frac{(a_1 + a_2)}{2} + o(1) \right)^2 2\ell_1 - (a_1 + o(1))^2 \ell_1 - (a_2 + o(1))^2 \ell_1 \right\} \\ = \quad & -\frac{d\ell_1}{4} \left((a_1 - a_2)^2 + o(1) \right) < 0 \, . \end{aligned}$$

Employing the mean value theorem yields

$$\int_{x_0-\ell_1}^{x_0+\ell_1} \{F(u_{new}) - F(u_0)\}\, e^x dx = -\int_{x_0-\ell_1}^{x_0+\ell_1} f(\tilde{u})(u_{new} - u_0)\, e^x dx$$

for some $\tilde{u}$ lying in between u_0 and u_{new}. Since

$u_{new} - u_0 = O(\ell_1)$,

$$|\int_{x_0-\ell_1}^{x_0+\ell_1} \{F(u_{new}) - F(u_0)\}\, e^x dx\,| \le \ell_1\, O(\ell_1),$$

which is negligible compared with the change in the gradient term of J.

Now turn to the nonlocal term of J. Since both $|\mathcal{L}_c u_0|$ and $|\mathcal{L}_c u_{new}|$ are bounded and $|u_{new} - u_0| = O(\ell_1)$,

$$|\frac{1}{2}\int_{x_0-\ell_1}^{x_0+\ell_1} \{u_{new}\,\mathcal{L}_c u_{new} - u_0\,\mathcal{L}_c u_0\}\, e^x dx\,| \leq \ell_1\, O(\ell_1)\ ,$$

which is also negligible compared with the change in the gradient term.

Therefore $J(u_{new}) < J(u_0)$ with $u_{new} \in \mathcal{A}$. This contradicts u_0 being a minimizer in $\mathcal{A}$.

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Theorem. Given $\beta \in (0, 1/2)$ and $\gamma < 4/(1-\beta)^2$, there is a $\hat{d} = \hat{d}(\gamma)$ such that if $d \leq \hat{d}$ then for some $c > 0$, (FN) possesses a traveling pulse solution (u_0, v_0). Moreover $u_0, v_0 \in C^\infty(\mathbf{R})$ and are exponentially decaying to 0 as $|x| \to \infty$.

Suppose that $u_0 \geq 0$ and oscillates infinite number of times near $-\infty$. Then $u_0 \geq 0$ on $(-\infty, z_3]$. For $x_1 < x_2 < z_3$,

$$\begin{aligned} w_1(x_1, x_2) &\equiv dc_0^2(u_0'(x_2) - u_0'(x_1) + u_0(x_2) - u_0(x_1)) \\ &= \int_{x_1}^{x_2} v_0 \, dx - \int_{x_1}^{x_2} f(u_0) \, dx \, , \end{aligned}$$

Then w_1 is uniformly bounded for any choice of x_1 and x_2.

$$\gamma \int_{x_1}^{x_2} v_0 \, dx - \int_{x_1}^{x_2} u_0 \, dx = w_2(x_1, x_2) \, .$$

$$w_1 - \frac{w_2}{\gamma} = \int_{x_1}^{x_2} \{\frac{u_0}{\gamma} - f(u_0)\} \, dx \geq m \int_{x_1}^{x_2} u_0 \, dx$$

for some positive constant m, since the graph $v = f(u)$ lies underneath the line $v = u/\gamma$ when $u \geq 0$. Thus

$$0 < \int_{-\infty}^{z_3} u_0 \, dx \leq M/m \, .$$

Remark

Suppose there exist $a_1 < b_1 \leq a_2 < b_2 \leq a_3 < b_3 \ldots$ in the interval $[x_0 - \ell, x_0]$ such that

$$\begin{cases} u_0 < \beta_2 & \text{on intervals } (a_i, b_i),\ i = 1, 2, \ldots\ , \\ u_0 = \beta_2 & \text{on } [x_0 - \ell, x_0] \setminus \cup_{i=1}^{\infty} (a_i, b_i)\ , \end{cases}$$

with both $a_i \to x_0^-$ and $b_i \to x_0^-$, then $u_0 \in C^1[x_0 - \ell, x_0)$ by Corner Lemma.